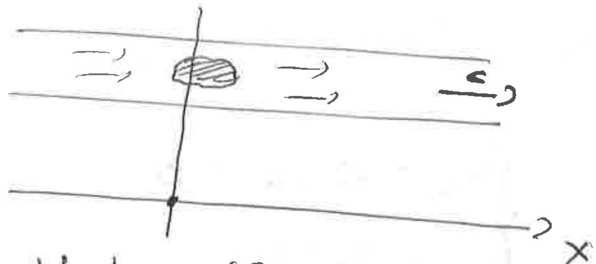
3) First order linear equations - transport eq. & conservation laws

3.1) Derivation of the transport eq.:

Let us suppose we have a pollutant on a river. The pollutant can only be transported by the flow, that we suppose having a velocity c on the right [if $c<0$ it goes to the left!]



Let $u(x,t)$ be the density of the pollution in the pt x at time t. Then, for any $x_1 < x_2$, $\forall t,s > 0$,

$$\int_{x_1}^{x_2} u(x,s)\,dx = \int_{x_1+ct}^{x_2+ct} u(x,s+t)\,dx.$$

By differentiating w.r.t. x_2 we get

$$u(x_2,s) = u(x_2+ct, s+t)$$

and by differentiating w.r.t. t, we get

$$\boxed{0 = c\,u_x + u_t}$$

$\downarrow$

the transport eq.

3.2) Solution of the transport eq.:

Even if the solution of the above eq. was, in a sense, already present in the derivation of the eq. itself, we would like to pretend we are facing the following pb for the first time:

$$\begin{cases} u_t + c u_x = 0 & \text{in } \mathbb{R} \times [0, +\infty), \\ u(x,0) = \underbrace{g(x)} & \text{in } \mathbb{R}. \end{cases}$$

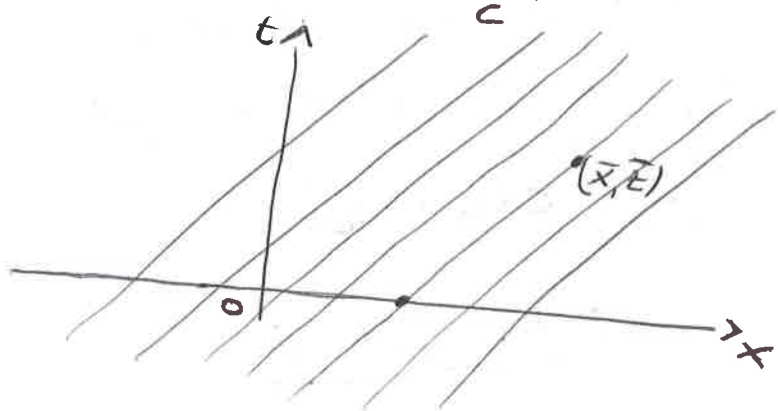
↳ the initial condition

• **idea**: the first equation can be written as:

$$(\partial_x u, \partial_t u) \cdot (c, 1) = \nabla u \cdot (c, 1) = 0$$

$$\Downarrow$$

$(x,t) \mapsto u(x,t)$ is constant along the lines with slope $\frac{1}{c}$.



These lines are called the characteristic lines.

→ what happen when $c \to 0$ and $c \to +\infty$?

→ The idea for finding the solution u at a pt $(\bar{x},\bar{t})$ is the following:

i) we find the line with slope $\frac{1}{c}$ passing through $(\bar{x},\bar{t})$

ii) we find the pt $(x_0,0)$ lieing on that line

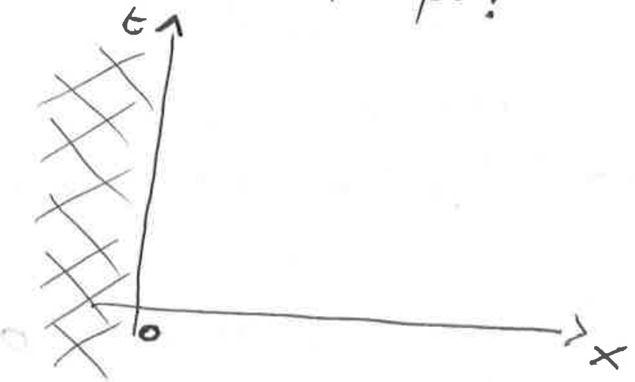
iii) $u(\bar{x},\bar{t}) = g(x_0)$

$$\Rightarrow \quad \boxed{u(x,t) = g(x-ct)}$$

→ the initial data g is <u>transported</u> along the characteristic lines!

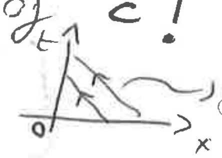
3.3) <u>Transport equation on a strip</u>:

Suppose we are interested in the transport eq. <u>only</u> in a region $[0,+\infty)$. Do we have enough information to solve the pb?



It depends on the sign of c!

- if $c<0$: yes



- if $c>0$: we need to assign u on $\underline{\{x=0\}}$

3.4) Variable coefficients homogeneous pb:

We would like to generalize the above pb to a more general situation, where the coefficients in front of u_t and u_x are variable. Namely, we would like to consider the following pb:

$$(*)\quad \begin{cases} a(x,t)\,u_t(x,t) + b(x,t)\,u_x(x,t) = 0 & \text{in } \mathbb{R}\times(0,+\infty), \\ u(x,0) = g(x) & \text{in } \mathbb{R}. \end{cases}$$

• <u>idea</u>: as before, we would like to find a **curve** $s \mapsto z(s) = (z_1(s), z_2(s)) \in \mathbb{R}^2$ on which the function u is constant, i.e.

$$s \mapsto u(z(s)) \equiv \text{constant}.$$

By differentiating this expression w.r.t. s, we get

here, with $\dot{z}_i(\bar{s}) := \frac{dz_i}{ds}(\bar{s})$

$$\dot{z}_1(s)\,u_t(z(s)) + \dot{z}_2(s)\,u_x(z(s)) = 0$$

Thus, <u>if</u> z satisfies

$$(CS)\quad \begin{cases} \dot{z}_1(s) = a(z(s)) \\ \dot{z}_2(s) = b(z(s)) \end{cases}$$

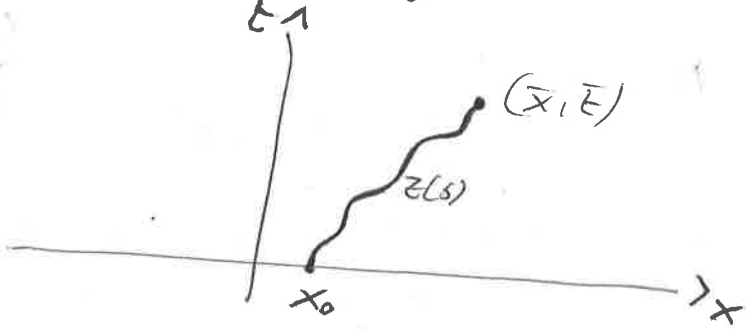
we are done!

So, to solve (*) we proceed as follows:

i) fix $(\bar{x}, \bar{t})$

ii) find the solution of (CS) passing through $(\bar{x}, \bar{t})$; this will pass through $(x_0, 0)$

iii) $u(\bar{x}, \bar{t}) := g(x_0)$



$\Rightarrow$ Problems: 1) given $(\bar{x}, \bar{t}) \in \mathbb{R}^2$ there is the possibility that no solution of (CS) passes through it, or that more than one does

2) there can be no pt $(x_0, 0)$

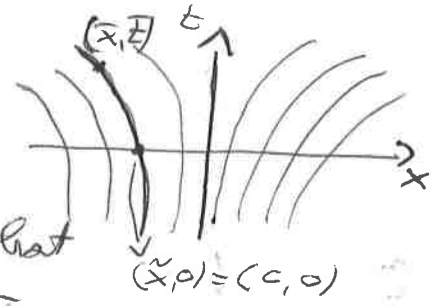
• Notice: • in practice, it is easier to solve

$$\boxed{\dot{p}(t) = \frac{b(p(t),t)}{a(p(t),t)}},$$

• if $a(x,t) \neq 0$. In this case, if we further assume $|b(x,0)| \geq C > 0$, there exists a unique solution, i.e., we have that $\forall (x,t) \in \mathbb{R} \times (0, \infty)$ we can find a unique solution of (CS) connecting (x,t) to a pt $(\bar{x}, 0)$.

• in the case where $|a(x,t)| \neq 0$ but $b(x,t)$ can vanish, we can have regions of the plane where no characteristic passes, or the characteristics do not meet the x-axes.

· Example: 1)
$$\begin{cases} u_t + x u_x = 0 & \text{in } \mathbb{R}\times(0,+\infty), \\ u(x,0) = g(x) & \text{in } \mathbb{R}. \end{cases}$$



$\Rightarrow$ $\dot{p}(t) = p(t) \rightsquigarrow p(t) = ce^{t}$

$\to$ given $(\bar{x},\bar{t}) \in \mathbb{R}^2$, we have that
$$\bar{x} = ce^{\bar{t}} \Rightarrow c = \bar{x}e^{-\bar{t}}$$
Moreover, for $t=0$, we get $p(0) = c$
$$\Downarrow$$
The solution is:
$$\boxed{u(x,t) = g(xe^{-t})}$$

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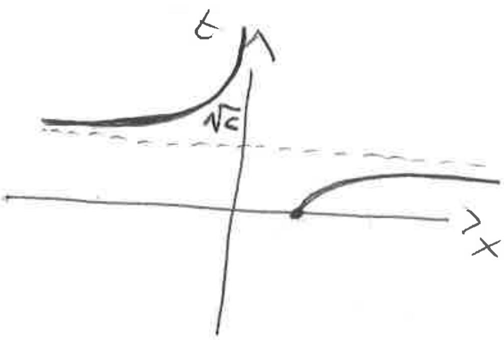
2)
$$\begin{cases} u_t + 2tx^2 u_x = 0 & \text{in } \mathbb{R}\times(0,+\infty), \\ u(x,0) = g(x) & \text{in } \mathbb{R}. \end{cases}$$

$\Rightarrow$ $\dfrac{dp}{dt} = 2tp^2 \underset{\text{separation of variables}}{\rightsquigarrow} \dfrac{dp}{p^2} = 2t\,dt$
$$\Downarrow$$
$$\Leftarrow \quad -\frac{1}{p} = t^2 - c$$



$$p(t) = \frac{1}{c-t^2}$$

Then: • fixed $(\bar{x},\bar{t})$, we have: $c = \bar{t}^2 + \frac{1}{\bar{x}}$

• for $t=0$ we get $p(0) = \frac{1}{c}$

Thus, the solution is:

$$\boxed{u(x,t) = g\left(\frac{x}{1+t^2x}\right)}$$

--> So, the basic idea is to reduce the **PDE** to some **ODEs**.
This strategy is called characteristic method, and the idea is to **transport** the initial data along characteristic curves.

3.5) Constant coefficient inhomogeneous pb:

Let us consider the pb of a pollutant transported by a river, but with also a source function:

$$\begin{cases} u_t + c u_x = f(x,t) & \text{in } \mathbb{R}\times(0,+\infty), \\ u(x,0) = g(x) & \text{in } \mathbb{R}. \end{cases}$$

Notice that the left hand-side is **linear** in u.

This allows us to split the pb in two subproblems:

$$(P_1) \quad \begin{cases} w_t + c w_x = 0 & \text{in } \mathbb{R}\times(0,+\infty), \\ w(x,0) = g(x) & \text{in } \mathbb{R}. \end{cases}$$

$$(P_2) \quad \begin{cases} z_t + c z_x = f(x,t) & \text{in } \mathbb{R}\times(0,+\infty), \\ z(x,0) = 0 & \text{in } \mathbb{R}. \end{cases}$$

$$\Longrightarrow \boxed{u = w + z}$$

We already know how to solve (P_1).
So, let us concentrate on (P_2).

a) First strategy :

As before, we notice that the lhs is the derivative w.r.t. s of the function $s \mapsto z(x+sc, t+s)$.
So we can rewrite the first eq. as:

$$\dot{p}(s) = f(x+sc, t+s),$$

getting:

$$z(x,t) = p(0) = p(-t) + (p(0) - p(-t))$$

$$= z(x-ct, 0) + \int_{-t}^{0} f(x+sc, t+s)\, ds$$

$$\underset{t+s=\tau}{=} 0 + \int_0^t f(x+(\tau-t)c, \tau)\, d\tau$$

$$\Rightarrow \boxed{u(x,t) = g(x-ct) + \int_0^t f(x+(s-t)c, s)\, ds}$$

b) <u>Second strategy – the Duhamel's principle</u>

We would like to obtain the solution of the problem

$$\begin{cases} u_t + c\,u_x = f(x,t) & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = g(x) & \text{in } \mathbb{R}, \end{cases}$$

by means of a "heuristic" way of thinking.
It is possible to give the following meaning to the above equation: let us consider a river moving with speed c. Assume that, at the time t in the point x we drop a number $f(x,t)$ of balls [assume, for the moment, that it is positive], and they all start moving with velocity c.
The question is: what is the number of balls that I see at the point $\bar{x}$ at time $\bar{t}$?
Let us denote this number by $u(\bar{x},\bar{t})$.
Well, it is clear that we have to sum all the balls that were able to reach the point $\bar{x}$ at time $\bar{t}$ by moving with speed c.
Clearly, we don't care about events that happen for times $s > \bar{t}$, but only what happens in the time interval $[0,\bar{t}]$. The question is: if at time s we drop a ball at a point $\tilde{x}$, what $\tilde{x}$ has to be in order to find that particle at time $\bar{t}$ at the point $\bar{x}$?

Since the particle will move with speed c, we have that:

$$\underbrace{x}_{\text{final position}} = \underbrace{\tilde{x}}_{\text{initial position}} + \underbrace{c}_{\text{speed}}\underbrace{(t-s)}_{\text{amount of time the particle moves}}$$

That is:

$$\tilde{x} = x - c(t-s).$$

Since every particle moves independently from all other particles [by the linearity of the equation], we just have to sum all the contributions for times $s \in [0, \bar{t}]$, that is,

$$u(\bar{x}, \bar{t}) = \underbrace{g(x-ct)}_{\substack{\text{when} \\ s=0}} + \int_0^{\bar{t}} \underbrace{f(x-c(t-s), s)}_{\substack{\text{number of particles that,} \\ \text{at time } s, \text{ we drop at the} \\ \text{point } x-c(t-s).}} \, ds$$

Notice that the same argument applies also when f is negative, i.e., we take away particles.